EXPLICIT HOPF-GALOIS DESCRIPTION OF $SL_{e^{\frac{2\pi i}{3}}}(2)$-INDUCED FROBENIUS HOMOMORPHISMS

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Abstract

The exact sequence of "coordinate-ring" Hopf algebras $A(SL(2,\mathbb{C}))\xrightarrow{Fr}A(SL_q(2))\rightarrow A(F)$ determined by the Frobenius map Fr, and the same way obtained exact sequence of (quantum) Borel subgroups, are studied when q is a cubic root of unity. An $A(SL(2,\mathbb{C}))$-linear splitting of $A(SL_q(2))$ making $A(SL(2,\mathbb{C}))$ a direct summand of $A(SL_q(2))$ is constructed and used to prove that $A(SL_q(2))$ is a faithfully flat $A(F)$-Galois extension of $A(SL(2,\mathbb{C}))$. A cocycle and coaction determining the bicrossed-product structure of the upper-triangular (Borel) quantum subgroup of $A(SL_q(2))$ are computed explicitly.

Introduction

This work was inspired by the final remark in [C-A96] pointing to the possible importance of quantum-group Frobenius homomorphisms in understanding the (quantum) symmetry of the Standard Model. We focus our attention on the cubic root of unity because it is the simplest non-trivial odd case and because, as advocated by A. Connes, it might be the "cubic symmetry" that is to succeed the supersymmetry in physics.

In the present study of short exact sequences of quantum groups we adopt the functions-on-group point of view, which is dual to the universal-enveloping-algebra approach (see Paragraph 8.17 in [L-G91]). It is known [A-N96, MS] that Frobenius mappings at primitive odd roots of unity allow us to view $A(SL_q(2))$ as a faithfully flat Hopf-Galois extension of $A(SL(2,\mathbb{C}))$. The main contribution of this paper is a construction of an $A(SL(2,\mathbb{C}))$-linear splitting of $A(SL_q(2))$ making $A(SL(2,\mathbb{C}))$ a direct summand of $A(SL_q(2))$, and the computation of the cocycle-bicrossed-product structure of the analogous quantum extension of the upper-triangular (Borel) subgroup of $SL(2,\mathbb{C})$. With the aim of attracting a diverse readership, we write this article in a relatively self-contained down-to-earth manner. We hope that, by exemplifying certain concepts in a very tangible way, this note can serve as an invitation to further study.

In the next two sections, we establish the basic language of this work and review appropriate modifications of known general results that we apply later to compute examples.

In Section 3, we reduce the task of computing the $A(F)$-coinvariants of $A(SL_q(2))$ to finding a certain $A(SL(2,\mathbb{C}))$-homomorphism. Just as Hopf-Galois extensions generalise to a great

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extent the concept of a principal bundle, this homomorphism generalises the notion of a section of a bundle. Thus we derive an alternative proof that $A(SL_q(2))$ is a faithfully flat Hopf-Galois extension of $A(SL(2,\mathbb{C}))$.

Section 4 and Section 5 are devoted to the study of the same kind Frobenius homomorphisms in the Borel and Cartan cases. As the Hopf algebra $P_+ := A(SL_q(2))/\langle c\rangle$ is pointed, we can conclude that P_+ is a cleft Hopf-Galois extension of $B_+ := A(SL(2,\mathbb{C}))/\langle \bar{c}\rangle$. We construct a family of cleaving maps $A(F)/\langle \tilde{c}\rangle =: H_+ \stackrel{\Phi_\nu}{\to} P_+$, calculate an associated cocycle and weak coaction, and prove that P_+ has a non-trivial bicrossed-product structure. Our construction works for any primitive odd root of unity. The Cartan case (the off-diagonal generators put to zero) is commutative and follows closely the Borel case pattern.

For the sake of completeness, in the final two sections we determine the integrals in and on $A(F)$, prove the non-existence of the Haar measure on F, and show that the natural $A(F)$-coinvariants of the polynomial algebra of the quantum plane at the cubic root of unity form an algebra isomorphic with the algebra of polynomials on $\mathbb{C}^2$. We also present corepresentations of $A(F)$.

Throughout this paper we use Sweedler's notation (with the summation symbol suppressed) for the coproduct ($\Delta\ h = h_{(1)} \otimes h_{(2)}$) and right coaction ($\Delta_R\ p = p_{(0)} \otimes p_{(1)}$). The unadorned tensor product stands for the tensor product over a field k. (In the examples $k = \mathbb{C}$.) The counit and antipode are denoted by ε and S respectively, and m is used to signify the multiplication in an algebra. By the convolution product of two linear maps we understand $f*g := m\circ(f\otimes g)\circ\Delta$, $(f*g)(h) = f(h_{(1)})g(h_{(2)})$. The convolution inverse of f is denoted by f^{-1} and defined by $f*f^{-1} = \varepsilon = f^{-1}*f$. We use δ_{kl} to denote the Kronecker delta.

1 Preliminaries

We begin by recalling basic definitions.

Definition 1.1 *Let H be a Hopf algebra, P be a right H-comodule algebra, and $B := P^{coH} := \{p \in P \mid \Delta_R\ p = p \otimes 1\}$ the subalgebra of right coinvariants. We say that P is a (right)* Hopf-Galois extension *(or H-Galois extension) of B iff the canonical left P-module right H-comodule map $can := (m_P \otimes id) \circ (id \otimes_B \Delta_R)\ :\ P \otimes_B P \longrightarrow P \otimes H$ is bijective.*

In what follows, we will use only right Hopf-Galois extensions, and skip writing "right" for brevity.

Definition 1.2 *We say that P is a faithfully flat H-Galois extension of B iff P is faithfully flat as a right and left B-module. (For a comprehensive review of the concept of faithful flatness see [B-N72].)*

Definition 1.3 *An H-Galois extension is called* cleft *iff there exists a convolution invertible linear map $\Phi : H \to P$ satisfying $\Delta_R \circ \Phi = (\Phi \otimes id) \circ \Delta$. We call Φ a cleaving map of P.*

Note that, in general, Φ is *not* uniquely determined by its defining conditions. Observe also that a cleaving map can always be normalised to be unital. Indeed, let $\tilde{\Phi}$ be a cleaving map, and $\tilde{\Phi}(1) := b$. By the colinearity, we have that $b \in B$, and the convolution invertibility entails that

b is invertible. Also, $b^{-1}\otimes 1 = b^{-1}\Delta_R(bb^{-1}) = b^{-1}b\Delta_R(b^{-1}) = \Delta_R(b^{-1})$. It is straightforward to check that $\Phi := b^{-1}\tilde{\Phi}$ is right colinear, convolution invertible and unital. Hence, without the loss of generality, we assume Φ to be unital for the rest of this paper. Let us also remark that a cleaving map is necessarily injective:

$$\big(m_P\circ(m_P\otimes id)\circ(id\otimes\Phi^{-1}\otimes id)\circ(id\otimes\Delta)\circ\Delta_R\circ\Phi\big)(h) = \Phi(h_{(1)})\Phi^{-1}(h_{(2)})h_{(3)} = h, \quad \forall\, h\in H.$$

Definition 1.4 ([PW91]) *A sequence of Hopf algebras (and Hopf algebra maps)* $B \xrightarrow{j} P \xrightarrow{\pi} H$ *is called* exact *iff* j *is injective and* π *is the canonical surjection on* $H = P/Pj(B^+)P$, *where* B^+ *denotes the augmentation ideal of* B *(kernel of the counit map).*

When no confusion arises regarding the considered class of "functions" on quantum groups, one can use the above definition to define exact sequences of quantum groups (see (1.6a) in [PW91]). In particular, we can view F as a finite quantum group. Further sophistication of the concept of a short exact sequence of quantum groups comes with the following definition (cf. [AD95, p.23]):

Definition 1.5 (p.3338 in [S-H93]) *An exact sequence of Hopf algebras* $B \xrightarrow{j} P \xrightarrow{\pi} H$ *is called* strictly exact *iff* P *is right faithfully flat over* $j(B)$, *and* $j(B)$ *is a normal Hopf subalgebra of* P, *i.e.,* $\big(p_{(1)}j(B)S(p_{(2)}) \cup S(p_{(1)})j(B)p_{(2)}\big) \subseteq j(B)$ *for any* $p\in P$.

See [M-A94, Section 5] for short exact sequences of finite dimensional Hopf algebras.

Remark 1.6 Exact sequences of Hopf algebras should not be confused with exact sequences of vector spaces: The exact sequence of groups $\mathbb{Z}_3 \to \mathbb{Z}_6 \to \mathbb{Z}_6/\mathbb{Z}_3 \cong \mathbb{Z}_2$ yields (by duality) an exact sequence of Hopf algebras which is *not* an exact sequence in the category of vector spaces (or algebras). $\diamond$

Let us now provide a modification of Remark 1.2(1) in [S-H92] that allows us to avoid directly verifying the faithful flatness condition in the proof of Proposition 3.4. We replace the faithful flatness assumption by assuming the existence of a certain homomorphism. Its existence in the case described in Proposition 3.4 is proved through a calculation (Lemma 3.5).

Lemma 1.7 *Let* P *be a right* H*-comodule algebra and* C *a* subalgebra *of* P^{coH} *such that the map* $\psi : P\otimes_C P \ni p\otimes_C p' \mapsto pp'_{(0)}\otimes p'_{(1)} \in P\otimes H$ *is bijective, and such that there exists a unital right* C*-linear homomorphism* $s : P \to C$ *(cf. Definition A.4 in [H-P96]). Then* $C = P^{coH}$, *and* P *is an* H*-Galois extension of* C.

Proof. Note first that the map ψ is well defined due to the assumption $C \subseteq P^{coH}$. Now, let x be an arbitrary element of P^{coH}. Then

$$1\otimes_C x = \psi^{-1}(\psi(1\otimes_C x)) = \psi^{-1}(x\otimes 1) = x\otimes_C 1\,. \tag{1.1}$$

On the other hand, we know from Proposition 2.5 of [CQ95] that $P\otimes_C (P/C)$ is isomorphic with $\mathrm{Ker}(m_p : P\otimes_C P \to P)$. In particular, this isomorphism sends $1\otimes_C x - x\otimes_C 1$ to $1\otimes_C [x]_C \in P\otimes_C (P/C)$. Remembering (1.1) and applying first $s\otimes_C id$ and then the multiplication map to $1\otimes_C [x]_C$, we obtain $[x]_C = 0$, i.e. $x\in C$, as needed. $\square$

Remark 1.8 Observe that the assumption of the existence of a unital right C-linear homomorphism $s : P \to C$ can be replaced by the assumption that P/C is flat as a left C-module. Indeed, we could then view $C \otimes_C (P/C)$ as a submodule of $P \otimes_C (P/C)$, and consequently $1 \otimes_C [x]_C$ as an element of the former. Now one could directly apply the multiplication map to $1 \otimes_C [x]_C$ and conclude the proof as before. ◇

It is well known that cleft Hopf-Galois extensions and crossed products are equivalent notions. Once we have a cleaving map Φ, we can determine the cocycle and cocycle action that define the crossed product structure (see [BCM86, Section 4], [M-S95, Definition 6.3.1]) from the following formulas respectively [S-H94, p.273]:

$$h \triangleright_\Phi b := \Phi(h_{(1)}) b \Phi^{-1}(h_{(2)}) \in P^{coH} \tag{1.2}$$

$$\sigma_\Phi(h \otimes l) := \Phi(h_{(1)})\Phi(l_{(1)})\Phi^{-1}(h_{(2)}l_{(2)}) \in P^{coH} \ , \tag{1.3}$$

where $h, l \in H,\ b \in P^{coH}$. On the other hand, with the help of Φ we can construct a unital left B-module homomorphism $s_\Phi : P \to B$ by the formula

$$s_\Phi := m \circ (id \otimes \Phi^{-1}) \circ \Delta_R \, . \tag{1.4}$$

The homomorphism s_Φ generalises the notion of a section of a principal bundle just as Φ generalises the concept of a trivialisation of a principal bundle (see the end of Section 4 here and Remark 2.6 in [H-P96]). The following straightforward-to-prove lemma allows one to compute σ_Φ by taking advantage of s_Φ. It seems to be a more convenient way of calculating σ_Φ whenever $\Delta \otimes \Delta$ is more complicated than Δ_R. We will use it to compute a cocycle of the cleft extension describing an exact sequence of (quantum) Borel subgroups.

Lemma 1.9 (cf. Lemma 2.5 in [H-P96]) *Let P be a cleft H-Galois extension of B and Φ a cleaving map. Then $\sigma_\Phi = s_\Phi \circ m \circ (\Phi \otimes \Phi)$.*

Finally, let us observe that with the help of the translation map (e.g., see [B-T96])

$$\tau : H \to P \otimes_B P,\ \tau(h) := can^{-1}(1 \otimes h) =: h^{(1)} \otimes_B h^{(2)}$$

(summation suppressed), we can solve formula (1.4) for Φ. Indeed,

$$(id *_\tau s_\Phi)(h) := h^{(1)} s_\Phi(h^{(2)}) = h^{(1)} h^{(2)}{}_{(0)} \Phi^{-1}(h^{(2)}{}_{(1)}) = \big(m \circ (id \otimes \Phi^{-1}) \circ can\big)(h^{(1)} \otimes_B h^{(2)}) = \Phi^{-1}(h),$$

whence $\Phi = (id *_\tau s_\Phi)^{-1}$.

2 Principal homogenous extensions

Let P be a Hopf algebra and a (P/I)-Galois extension of B for the coaction

$$\Delta_R := (id \otimes \pi) \circ \Delta, \quad P \xrightarrow{\pi} P/I,$$

where I is a Hopf ideal of P. Then we call P is a *principal homogenous extension* of B. First we recall a theorem[1] which shows the structure of the Hopf ideal I.

[1] We owe it to Peter Schauenburg.

Theorem 2.1 (cf. Lemma 5.2 in [BM93]) *Let P be a Hopf algebra and I a Hopf ideal of P. Then P is a (P/I)-principal homogenous extension of B* if and only if *$I = B^+P$, where $B := P^{co(P/I)}$, $B^+ := B \cap \operatorname{Ker} \varepsilon$.*

Proof. Assume first that $I = B^+P$. Taking advantage of (2.7), for any $b \in B^+$, $p \in P$, we have:

$$S(b_{(1)}p_{(1)}) \otimes_B b_{(2)}p_{(2)} = S(b_{(1)}p_{(1)})b_{(2)} \otimes_B p_{(2)} = S(p_{(1)})\varepsilon(b) \otimes_B p_{(2)} = 0. \tag{2.5}$$

Hence we have a well-defined map $\wp : P \otimes (P/I) \to P \otimes_B P$, $\wp(p \otimes [p']_I) := pS(p'_{(1)}) \otimes_B p'_{(2)}$. It is straightforward to verify that $\wp$ is the inverse of the canonical map *can*. Consequently, P is a Hopf-Galois extension.

To show the converse, let us first prove the following:

Lemma 2.2 *Let P, I and B be as above. Then $B \subseteq P$ is a (P/I)-Galois extension* if and only if *$\big(\pi_B \circ (S \otimes id) \circ \Delta\big)(I) = 0$, where $\pi_B : P \otimes P \to P \otimes_B P$ is the canonical surjection.*

Proof. If P is a (P/I)-Galois extension of B, then we have the following short exact sequence (see the proof of Proposition 1.6 in [H-P96]):

$$0 \to P(\Omega^1 B)P \hookrightarrow P \otimes P \overset{T_R}{\to} P \otimes P/I \to 0. \tag{2.6}$$

Here $\Omega^1 B := \operatorname{Ker}(m : B \otimes B \to B)$ and $T_R = (m \otimes \pi) \circ (id \otimes \Delta)$. One can check that $\big(T_R \circ (S \otimes id) \circ \Delta\big)(I) = 0$. Hence, it follows from the exactness of (2.6) that $\big((S \otimes id) \circ \Delta\big)(I) \subseteq P(\Omega^1 B)P$. Consequently, $\big(\pi_B \circ (S \otimes id) \circ \Delta\big)(I) = 0$ due to the exactness of the sequence

$$0 \to P(\Omega^1 B)P \hookrightarrow P \otimes P \overset{\pi_B}{\to} P \otimes_B P \to 0\,.$$

To prove the converse, one can proceed as in the considerations preceding this lemma. $\square$

Corollary 2.3 *Let $B \subseteq P$ be a (P/I)-Galois extension as above. Then the translation map is given by the formula: $\tau([p]_I) := S(p_{(1)}) \otimes_B p_{(2)}$.*

Assume now that P is a (P/I)-Galois extension of B. It follows from the above corollary and (2.5) that $\tau([B^+P]_I) = 0$. Hence, by the injectivity of τ, we have $B^+P \subseteq I$. Furthermore, we have a well-defined map $can' : P \otimes_B P \to P \otimes (P/B^+P)$, $p \otimes_B p' \mapsto pp'_{(1)} \otimes [p'_{(2)}]_{B^+P}$. Indeed, taking again advantage of (2.7), we obtain

$$\begin{aligned} p \otimes bp' &\mapsto pb_{(1)}p'_{(1)} \otimes [(b_{(2)} - \varepsilon(b_{(2)}))p'_{(2)} + \varepsilon(b_{(2)})p'_{(2)}]_{B^+P} \\ &= pb_{(1)}p'_{(1)} \otimes \varepsilon(b_{(2)})[p'_{(2)}]_{B^+P} \\ &= pbp'_{(1)} \otimes [p'_{(2)}]_{B^+P}\,, \end{aligned}$$

and $pb \otimes p' \mapsto pbp'_{(1)} \otimes [p'_{(2)}]_{B^+P}$. Reasoning as in the first part of the proof, we can conclude that can' is bijective. We have the following commutative diagram:

$$\begin{tikzcd} P \otimes_B P \arrow[r, "can'"] \arrow[d, "id"'] & P \otimes (P/B^+P) \arrow[d, "id\otimes\ell"] \\ P \otimes_B P \arrow[r, "can"] & P \otimes (P/I)\,, \end{tikzcd}$$

where $\ell([p]_{B^+P}) := [p]_I$. (Recall that we have already showed that $B^+P \subseteq I$, so that ℓ is well defined.) It follows from the commutativity of the diagram that $id \otimes \ell$ is bijective. In particular, we have that ℓ is injective, and therefore $I \subseteq B^+P$, as needed. ■

Let us now prove the following left-sided version of a result by Y.Doi and A.Masuoka (see [MD92] or [M-A94, Proposition 3.8]):

Theorem 2.4 *Let P be a (P/I)-principal homogenous extension of B. Then P is cleft* if and only if *there exists a convolution invertible left B-module homomorphism* $\Psi : P \to B$.

Proof. Assume first that P is cleft. Let Φ be a cleaving map. Then $\Psi := s_\Phi$ (see (1.4)) is left B-linear. Moreover, it can be directly verified that $\Psi^{-1} : P \to B,\ \Psi^{-1}(p) := \Phi(\pi(p_{(1)}))S(p_{(2)})$, (see [AD95, Definition 3.2.13(3)]) is the convolution inverse of Ψ.

Conversely, assume that we have $\Psi : P \to B$ with the required properties. To define Φ in terms of Ψ, first we need to derive certain property of Ψ^{-1}.

Lemma 2.5 *Let $\Psi : P \to B$ be a homomorphism as described in Theorem 2.4. Then $\Psi^{-1}(b_{(1)}p)b_{(2)} = \varepsilon(b)\Psi^{-1}(p)$ for any $b \in B,\ p \in P$.*

Proof. Note first that $b \in B$ implies $b_{(1)} \otimes b_{(2)} \in P \otimes B$. Indeed,

$$\begin{aligned}(id \otimes \Delta_R)(b_{(1)} \otimes b_{(2)}) &= \big((id \otimes id \otimes \pi) \circ (id \otimes \Delta) \circ \Delta\big)(b) \\ &= \big((id \otimes id \otimes \pi) \circ (\Delta \otimes id) \circ \Delta\big)(b) \\ &= \big((\Delta \otimes id) \circ \Delta_R\big)(b) \\ &= b_{(1)} \otimes b_{(2)} \otimes 1. \end{aligned} \tag{2.7}$$

Taking advantage of this fact, for any $b \in B,\ p \in P$, we obtain

$$\begin{aligned}\varepsilon(b)\Psi^{-1}(p) &= \Psi^{-1}(b_{(1)}p_{(1)})\Psi(b_{(2)}p_{(2)})\Psi^{-1}(p_{(3)}) \\ &= \Psi^{-1}(b_{(1)}p_{(1)})b_{(2)}\Psi(p_{(2)})\Psi^{-1}(p_{(3)}) \\ &= \Psi^{-1}(b_{(1)}p)b_{(2)}\,,\end{aligned}$$

as claimed. □

On the other hand, we know (see Theorem 2.1) that, since P is a (P/I)-Galois extension, $I = B^+P$. Furthermore, with the help of Lemma 2.5, we can directly show that $(\Psi^{-1} * id)(B^+P) = 0$. Hence we have a well-defined map $\Phi : P/I \to P,\ \Phi(\pi(p)) := (\Psi^{-1} * id)(p) = \Psi^{-1}(p_{(1)})p_{(2)}$. We also have:

$$(\Delta_R \circ \Phi \circ \pi)(p) = (\Psi^{-1}(p_{(1)}) \otimes 1)(id \otimes \pi)(\Delta p_{(2)}) = \Psi^{-1}(p_{(1)})p_{(2)} \otimes \pi(p_{(3)}) = \Phi(\pi(p_{(1)})) \otimes \pi(p_{(2)})\,,$$

i.e., Φ is colinear. As expected from the general discussion in the previous section, the formula for the convolution inverse of Φ is $\Phi^{-1} = id *_\tau \Psi$. In our case we know that the formula for the translation map is $\tau(\pi(p)) = S(p_{(1)}) \otimes_B p_{(2)}$ (see [S-H92, p.294] and Corollary 2.3). Thus we obtain: $\Phi^{-1}(\pi(p)) = S(p_{(1)})\Psi(p_{(2)})$. It can be directly checked that Φ^{-1} is indeed the convolution inverse of Φ. ■

In the spirit of [AD95, p.47], we say that P is *cocleft* iff there exists a convolution invertible left B-module map (retraction) $\Psi : P \to B$. (See [MD92, Definition 2.2] for the right-sided version.) We call Ψ a cocleaving map.

Lemma 2.6 *Let P be P/I-principal homogenous extension of B. Any cleaving and any cocleaving map of such an extension can always be normalised to be both unital and counital.*

Proof. We already know from the previous section that a cleaving map can always be made unital. Similarly, for any cocleaving map $\tilde{\Psi} : P \to B$, the map defined by $\check{\Psi}(p) = \tilde{\Psi}(p)\tilde{\Psi}(1)^{-1}$ is a unital cocleaving map. Here the invertibility of $\tilde{\Psi}(1)$ follows from the convolution invertibility of $\tilde{\Psi}$, and $\check{\Psi}^{-1}(p) = \tilde{\Psi}(1)\tilde{\Psi}^{-1}(p)$ is the formula for the convolution inverse of $\check{\Psi}$. We can describe $\check{\Psi}$ as the composite mapping:

$$P \xrightarrow{id \otimes 1} P \otimes k \xrightarrow{\tilde{\Psi} \otimes \tilde{\Psi}(.)^{-1}} B \otimes B \xrightarrow{m_B} B.$$

Formally "dualising" this sequence and exchanging factors in the tensor product one obtains:

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{(\varepsilon \circ f^{-1}) \otimes f} k \otimes P \xrightarrow{m} P.$$

This suggests that one can counitalise a unital cleaving map $\check{\Phi} : P/I \to P$ by the formula[2] $\Phi(h) = \varepsilon(\check{\Phi}^{-1}(h_{(1)}))\check{\Phi}(h_{(2)})$. Indeed, it is straightforward to check that the thus defined map is counital, unital, colinear and convolution invertible with the convolution inverse given by $\Phi^{-1}(h) = \check{\Phi}^{-1}(h_{(1)})\varepsilon(\check{\Phi}(h_{(2)}))$. To complete the proof it suffices to check that if $\check{\Psi} : P \to B$ is a unital cocleaving map, then the map defined by the formula $\Psi(p) = \check{\Psi}(p_{(1)})\varepsilon(\check{\Psi}^{-1}(p_{(2)}))$ is unital, counital and cocleaving. The first two properties are immediate. Furthermore, it is straightforward to verify that $\Psi^{-1}(p) = \varepsilon(\check{\Psi}(p_{(1)}))\check{\Psi}^{-1}(p_{(2)})$ defines the convolution inverse of Ψ. It remains to make sure that Ψ is left B-linear. To this end, taking advantage of the fact that $b \in B \Rightarrow b_{(1)} \otimes b_{(2)} \in P \otimes B$ (see (2.7)) and Lemma 2.5, we compute:

$$\begin{aligned}
\Psi(bp) &= \check{\Psi}(b_{(1)}p_{(1)})\varepsilon(\check{\Psi}^{-1}(b_{(2)}p_{(2)})) \\
&= \check{\Psi}(b_{(1)}p_{(1)})\varepsilon(\check{\Psi}^{-1}(b_{(2)}p_{(2)})b_{(3)}) \\
&= \check{\Psi}(b_{(1)}p_{(1)})\varepsilon(\varepsilon(b_{(2)})\check{\Psi}^{-1}(p_{(2)})) \\
&= \check{\Psi}(bp_{(1)})\varepsilon(\check{\Psi}^{-1}(p_{(2)})) \\
&= b\Psi(p).
\end{aligned}$$

In the last step we used the assumption that $\check{\Psi}$ is left B-linear. □

Corollary 2.7 *Let P be P/I-principal homogenous extension of B. Then the following statements are equivalent:*

1. *P is cleft.*
2. *P is cocleft.*
3. *There exists a unital and counital convolution invertible right P/I-colinear map $\Phi : P/I \to P$.*
4. *There exists a unital and counital convolution invertible left B-linear map $\Psi : P \to B$.*

Also, we have a one-to-one correspondence between the unital-counital cleaving and the unital-counital cocleaving maps of P. The formula

$$\Phi \longmapsto \Psi := s_\Phi := m \circ (id \otimes \Phi^{-1}) \circ \Delta_R$$

defines the desired bijection. Its inverse is given by

$$\Psi \longmapsto \Phi, \quad \Phi(\pi(p)) := (\Psi^{-1} * id)(p) = \Psi^{-1}(p_{(1)})p_{(2)}\,.$$

[2] We owe this formula to a discussion with Tomasz Brzeziński.

Observe that our considerations are very similar to those on p.47 and p.50 in [AD95]. Here, however, we do not assume that the algebra of coinvariants is a Hopf algebra.

3 $A(SL_{e^{\frac{2\pi i}{3}}}(2))$ as a faithfully flat Hopf-Galois extension

Recall that $A(SL_q(2))$ is a complex Hopf algebra generated by 1, a, b, c, d, satisfying the following relations:

$$ab = qba\ , \quad ac = qca\ , \quad bd = qdb\ , \quad bc = cb\ , \quad cd = qdc\ ,$$
$$ad - da = (q - q^{-1})bc\ , \quad ad - qbc = da - q^{-1}bc = 1\ ,$$

where $q \in \mathbb{C} \setminus \{0\}$. The comultiplication Δ, counit ε, and antipode S of $A(SL_q(2))$ are defined by the following formulas:

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes 1 & b\otimes 1 \\ c\otimes 1 & d\otimes 1 \end{pmatrix}\begin{pmatrix} 1\otimes a & 1\otimes b \\ 1\otimes c & 1\otimes d \end{pmatrix},\ \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}.$$

Let us now establish some notation (e.g., see Section IV.2 in [K-Ch95]):

$$(k)_q := 1 + q + \ldots + q^{k-1} = \frac{q^k - 1}{q - 1}\ , \quad k \in \mathbb{Z},\ k > 0\ ;$$
$$(k)_q! := (1)_q(2)_q \ldots (k)_q = \frac{(q-1)(q^2-1)\ldots(q^k-1)}{(q-1)^k}\ , \quad (0)_q! := 1\ ;$$
$$\binom{k}{i}_q := \frac{(k)_q!}{(k-i)_q!(i)_q!}\ , \quad 0 \le i \le k\ .$$

The above defined q-binomial coefficients satisfy the following equality:

$$(u+v)^k = \sum_{l=0}^{k} \binom{k}{l}_q u^l v^{k-l},$$

where $uv = q^{-1}vu$. Now, if $(T_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\Delta T_{ij}^k = (T_{i1}\otimes T_{1j} + T_{i2}\otimes T_{2j})^k = \sum_{l=0}^{k} \binom{k}{l}_{q^{-2}} T_{i1}^l T_{i2}^{k-l} \otimes T_{1j}^l T_{2j}^{k-l}\ .$$

For the rest of this paper we put $q = e^{\frac{2\pi i}{3}}$. Obviously, we now have $q^{-2} = q$, and the comultiplication on the basis elements of $A(SL_q(2))$ (see Lemma 1.4 in [MMNNU91], Exercise 7 on p.90 in [K-Ch95]) is given by:

$$\Delta(a^p b^r c^s) = \sum_{\lambda,\mu,\nu=0}^{p,r,s} \binom{p}{\lambda}_q \binom{r}{\mu}_q \binom{s}{\nu}_q a^{p-\lambda} b^\lambda a^\mu b^{r-\mu} c^{s-\nu} d^\nu \otimes a^{p-\lambda} c^\lambda b^\mu d^{r-\mu} a^{s-\nu} c^\nu\ ,$$

$$\Delta(b^k c^l d^m) = \sum_{\lambda,\mu,\nu=0}^{k,l,m} \binom{k}{\lambda}_q \binom{l}{\mu}_q \binom{m}{\nu}_q a^\lambda b^{k-\lambda} c^{l-\mu} d^\mu c^\nu d^{m-\nu} \otimes b^\lambda d^{k-\lambda} a^{l-\mu} c^\mu b^\nu d^{m-\nu}\ , \qquad (3.8)$$

where m is a positive integer and p, r, s, k, l are non-negative integers.

Following Chapter 7 of [PW91] and Section 4.5 of [M-Yu91] (cf. Section 5 in [T-M92] and the end of Part I of [C-P94]), we take the Frobenius mapping

$$Fr : A(SL(2,\mathbb{C})) \ni \bar{T}_{ij} \longmapsto T^3_{ij} \in A(SL_q(2))\,,\quad i,j \in \{1,2\}, \tag{3.9}$$

to construct the exact sequence of Hopf algebras

$$A(SL(2,\mathbb{C})) \xrightarrow{Fr} A(SL_q(2)) \xrightarrow{\pi_F} A(F)\,. \tag{3.10}$$

Here $A(F) = A(SL_q(2))/\langle T^3_{ij} - \delta_{ij}\rangle\,,\quad i,j \in \{1,2\}$, and π_F is the canonical surjection. The following proposition determines a basis of $A(F)$ and shows that $A(F)$ is 27-dimensional.

Proposition 3.1 *Define* $\tilde{a} := \pi_F(a),\ \tilde{b} := \pi_F(b),\ \tilde{c} := \pi_F(c),\ \tilde{d} := \pi_F(d)$. *Then the set* $\{\tilde{a}^p\tilde{b}^r\tilde{c}^s\}_{p,r,s\in\{0,1,2\}}$ *is a basis of* $A(F)$*.*

Proof. Since $\tilde{d} = \tilde{a}^2(1 + q\tilde{b}\tilde{c})$, the monomials $\tilde{a}^p\tilde{b}^r\tilde{c}^s,\ p,r,s \in \{0,1,2\}$, span $A(F)$. Guided by the left action of $A(F)$ on itself, we define a 27-dimensional representation $\varrho : A(F) \to \mathrm{End}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ by the following formulas:

$$\begin{aligned}
\varrho(\tilde{a}) &= \mathbf{J} \otimes \mathbf{I_3} \otimes \mathbf{I_3}\ ,\\
\varrho(\tilde{b}) &= \mathbf{Q} \otimes \mathbf{N} \otimes \mathbf{I_3}\ ,\\
\varrho(\tilde{c}) &= \mathbf{Q} \otimes \mathbf{I_3} \otimes \mathbf{N}\ ,
\end{aligned}$$

where

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{pmatrix},\quad \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0\\ 0 & q^{-1} & 0\\ 0 & 0 & q^{-2} \end{pmatrix},\quad \mathbf{N} = \begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{pmatrix},\quad \mathbf{I_3} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

It is straightforward to check that ϱ is well defined. Assume now that $\sum_{p,r,s\in\{0,1,2\}} \alpha_{prs}\tilde{a}^p\tilde{b}^r\tilde{c}^s = 0$. Applying ϱ, we obtain

$$\sum_{p,r,s\in\{0,1,2\}} \alpha_{prs}\mathbf{J}^p\mathbf{Q}^{r+s} \otimes \mathbf{N}^r \otimes \mathbf{N}^s = 0\,. \tag{3.11}$$

On the other hand, let us consider the linear functionals

$$h^{klm} : M_3(\mathbb{C})^{\otimes 3} \to \mathbb{C}\,,\quad h^{klm}(A \otimes B \otimes C) := A_{k0}B_{l0}C_{m0}\,,\quad k,l,m \in \{0,1,2\}\,,$$

where we number the rows and columns of matrices by 0,1,2. From (3.11) we can conclude that

$$h^{klm}\left(\sum_{p,r,s\in\{0,1,2\}} \alpha_{prs}\mathbf{J}^p\mathbf{Q}^{r+s} \otimes \mathbf{N}^r \otimes \mathbf{N}^s\right) = 0,\quad \forall k,l,m \in \{0,1,2\}\ .$$

Consequently, since $h^{klm}(\mathbf{J}^p\mathbf{Q}^{r+s} \otimes \mathbf{N}^r \otimes \mathbf{N}^s) = \delta_{pk}\delta_{rl}\delta_{ms}$, we have that $\alpha_{prs} = 0$, for any p,r,s. Hence $\tilde{a}^p\tilde{b}^r\tilde{c}^s$ are linearly independent, as claimed. □

Corollary 3.2 (cf. Section 3 in [S-A97]) *The representation* $\varrho : A(F) \to \mathrm{End}(\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3)$ *defined above is faithful.*

Remark 3.3 Observe that we could equally well consider a representation with $\mathbf{Q}$ replaced by $\mathbf{Q}^{-1}$, $\mathbf{J}$ by $\mathbf{J}^t$ and $\mathbf{N}$ by $\mathbf{N}^t$, where $\quad ^t$ denotes the matrix transpose. $\diamond$

With the help of duality between functions-on-group and universal-enveloping algebra pictures, it can be shown ([A-N96, Proposition 3.4.5], [MS]) that $Fr\big(A(SL(2,\mathbb{C}))\big) = A(SL_q(2))^{A(F)}$. This can also be concluded from the fact that $A(SL_q(2))$ is Noetherian (see [K-Ch95, Theorem IV.4.1, Proposition I.8.2]) and the combination of [S-H93, Theorem 3.3], [S-H92, Remark 1.2(1)] and [S-H93, Remark 1.6(1)]. Note that Theorem 3.3 in [S-H93] establishes the faithful flatness of $A(SL_q(2))$ over $A(SL(2,\mathbb{C}))$. After identifying $A(SL(2,\mathbb{C}))$ with $A(SL_q(2))^{coA(F)}$, we can use Theorem 1.3 in [S-H94] (see [KT81]) to infer that $A(SL_q(2))$ is finitely generated projective over $A(SL(2,\mathbb{C}))$ (cf. [DL94, Proposition 1.7]). In what follows, we provide a direct proof which does not invoke the duality.

Proposition 3.4 *The algebra $A(SL(2,\mathbb{C}))$ of polynomial functions on $SL(2,\mathbb{C})$ is isomorphic (via the Frobenius map) to the subalgebra $A(SL_q(2))^{coA(F)}$ of all right coinvariants.*

Proof. The claim of the proposition follows immediately from the lemma below, [S-H92, Lemma 1.3(1)] and Lemma 1.7. (From these lemmas one can also conclude that $A(SL_q(2))$ is an $A(F)$-Galois extension of $Fr\big(A(SL(2,\mathbb{C}))\big)$.)

Lemma 3.5 *Let $p, r, s, k, l, m \in \mathbb{N}_0$, $m > 0$. The linear map $s : A(SL_q(2)) \to Fr\big(A(SL(2,\mathbb{C}))\big)$ defined by the formulas*

$$s(a^p b^r c^s) = \begin{cases} a^p b^r c^s & \text{when } p, r, s \text{ are divisible by } 3 \\ 0 & \text{otherwise}, \end{cases}$$
$$s(b^k c^l d^m) = \begin{cases} b^k c^l d^m & \text{when } k, l, m \text{ are divisible by } 3 \\ 0 & \text{otherwise}, \end{cases}$$

is a unital $Fr\big(A(SL(2,\mathbb{C}))\big)$-homomorphism.

Proof. The unitality is obvious. Next, as $Fr\big(A(SL(2,\mathbb{C}))\big)$ is a central subalgebra of $A(SL_q(2))$ (see Theorem 5.1.(a) in [M-Yu91]), the left and right $Fr\big(A(SL(2,\mathbb{C}))\big)$-module structure of $A(SL_q(2))$ coincide. Now, we want to show that $s(f\omega) = f s(\omega)$, for any $f \in Fr\big(A(SL(2,\mathbb{C}))\big)$ and $\omega \in A(SL_q(2))$. In terms of the basis of $A(SL_q(2))$, we have a natural decomposition $f = f^1 + f^2$, $\omega = \omega^1 + \omega^2$, where $f^1 = \sum f^1_{prs} a^{3p} b^{3r} c^{3s}$, $f^2 = \sum_{m>0} f^2_{klm} b^{3k} c^{3l} d^{3m}$, $\omega^1 = \sum \omega^1_{\alpha\beta\gamma} a^\alpha b^\beta c^\gamma$, $\omega^2 = \sum_{\nu>0} \omega^2_{\lambda\mu\nu} b^\lambda c^\mu d^\nu$. Unless otherwise specified, we sum here over non-negative integers. It is straightforward to see that $s(f^1\omega^1) = f^1 s(\omega^1)$ and $s(f^2\omega^2) = f^2 s(\omega^2)$. We will demonstrate that $s(f^2\omega^1) = f^2 s(\omega^1)$. We have:

$$\begin{aligned} f^2\omega^1 &= \sum_{m>0} f^2_{klm} b^{3k} c^{3l} d^{3m} \sum \omega^1_{\alpha\beta\gamma} a^\alpha b^\beta c^\gamma = \sum_{m>0} f^2_{klm} \omega^1_{\alpha\beta\gamma} d^{3m} a^\alpha b^{3k+\beta} c^{3l+\gamma} \\ &= \sum_{3m>\alpha} f^2_{klm} \omega^1_{\alpha\beta\gamma} d^{3m-\alpha} d^\alpha a^\alpha b^{3k+\beta} c^{3l+\gamma} + \sum_{0<3m\leq\alpha} f^2_{klm} \omega^1_{\alpha\beta\gamma} d^{3m} a^{3m} a^{\alpha-3m} b^{3k+\beta} c^{3l+\gamma} \\ &= \sum_{3m>\alpha} f^2_{klm} \omega^1_{\alpha\beta\gamma} d^{3m-\alpha} p_\alpha(b,c) b^{3k+\beta} c^{3l+\gamma} + \sum_{0<3m\leq\alpha} f^2_{klm} \omega^1_{\alpha\beta\gamma} a^{\alpha-3m} p_m(b^3,c^3) b^{3k+\beta} c^{3l+\gamma}. \end{aligned}$$

Here, due to the relation $da = 1 + q^{-1}bc$, the monomials $d^\alpha a^\alpha =: p_\alpha(b,c)$ and $d^{3m}a^{3m} =: p_m(b^3,c^3)$ are polynomials in b, c and b^3, c^3 respectively. Applying s yields:

$$
\begin{aligned}
s(f^2\omega^1) &= \sum_{m>\lambda} f^2_{klm}\omega^1_{3\lambda,\beta,\gamma} s\big(d^{3(m-\lambda)}p_{3\lambda}(b,c)b^{3k+\beta}c^{3l+\gamma}\big) \\
&+ \sum_{0<m\leq\lambda} f^2_{klm}\omega^1_{3\lambda,3\mu,3\nu} s\big(a^{3(\lambda-m)}p_m(b^3,c^3)b^{3(k+\mu)}c^{3(l+\nu)}\big) \\
&= \sum_{m>\lambda} f^2_{klm}\omega^1_{3\lambda,\beta,\gamma} s\big(d^{3\lambda}a^{3\lambda}b^{3k+\beta}c^{3l+\gamma}d^{3(m-\lambda)}\big) \\
&+ \sum_{0<m\leq\lambda} f^2_{klm}\omega^1_{3\lambda,3\mu,3\nu} a^{3(\lambda-m)}p_m(b^3,c^3)b^{3(k+\mu)}c^{3(l+\nu)} \\
&= \sum_{m>\lambda} f^2_{klm}\omega^1_{3\lambda,3\mu,3\nu} d^{3\lambda}a^{3\lambda}b^{3(k+\mu)}c^{3(l+\nu)}d^{3(m-\lambda)}\big) \\
&+ \sum_{0<m\leq\lambda} f^2_{klm}\omega^1_{3\lambda,3\mu,3\nu} a^{3(\lambda-m)}d^{3m}a^{3m}b^{3(k+\mu)}c^{3(l+\nu)}\,.
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
f^2 s(\omega^1) &= \sum_{m>0} f^2_{klm}b^{3k}c^{3l}d^{3m}\sum \omega^1_{3\lambda 3\mu 3\nu}a^{3\lambda}b^{3\mu}c^{3\nu} \\
&= \sum_{m>0} f^2_{klm}\omega^1_{3\lambda 3\mu 3\nu}d^{3m}a^{3\lambda}b^{3k+3\mu}c^{3l+3\nu} \\
&= \sum_{m>\lambda} f^2_{klm}\omega^1_{3\lambda 3\mu 3\nu}d^{3m-3\lambda}d^{3\lambda}a^{3\lambda}b^{3k+3\mu}c^{3l+3\nu} \\
&+ \sum_{0<m\leq\lambda} f^2_{klm}\omega^1_{3\lambda 3\mu 3\nu}d^{3m}a^{3m}a^{3\lambda-3m}b^{3k+3\mu}c^{3l+3\nu}.
\end{aligned}
$$

Hence $s(f^2\omega^1) = f^2 s(\omega^1)$, as needed. The remaining equality $s(f^1\omega^2) = f^1 s(\omega^2)$ can be proved in a similar manner. □

Note that it follows from the above lemma that $P = B \oplus (id - s)P$ as B-modules; cf. Lemma 3(3) in [R-D97].

Corollary 3.6 *$A(SL_q(2))$ is a faithfully flat $A(F)$-Galois extension of $Fr\big(A(SL(2,\mathbb{C}))\big)$.*

Proof. The fact that $A(SL_q(2))$ is an $A(F)$-Galois extension of $Fr\big(A(SL(2,\mathbb{C}))\big)$ can be inferred from the proof of Proposition 3.4.

Another way to see it is as follows: For any Hopf algebra P, the canonical map $P\otimes P \ni p\otimes p' \mapsto pp'_{(1)}\otimes p'_{(2)} \in P\otimes P$ is bijective. Consequently, for any Hopf ideal I of P, the canonical map $P\otimes_{P^{co(P/I)}} P \to P\otimes(P/I)$ is surjective. (Here we assume the natural right coaction $(id\otimes\pi)\circ\Delta : P \to P\otimes(P/I)$.) Now, since in our case we additionally have that $P/I = A(F)$ is finite dimensional, we can conclude that $A(SL_q(2))$ is an $A(F)$-Galois extension of $Fr\big(A(SL(2,\mathbb{C}))\big)$ by Proposition 3.4 and [S-H94, Theorem 1.3] (see [KT81]).

The faithful flatness of $A(SL_q(2))$ over $Fr\big(A(SL(2,\mathbb{C}))\big)$ follows from the commutativity of the latter and Corollary 1.5 in [S-H94] (see [KT81]). □

Remark 3.7 Note that just as the fact that $Fr\big(A(SL(2,\mathbb{C}))\big)$ is the space of all coinvariants implies that $A(SL_q(2))$ is faithfully flat over it, the faithful flatness of $A(SL_q(2))$ over $Fr\big(A(SL(2,\mathbb{C}))\big)$ entails, by virtue of [S-H92, Lemma 1.3(2)] (or the centrality of $Fr\big(A(SL(2,\mathbb{C}))\big)$ in $A(SL_q(2))$ and [S-H93, Remark 1.6(1)]), that $Fr\big(A(SL(2,\mathbb{C}))\big)$ is the space of all coinvariants. Therefore it suffices either to find all the coinvariants or prove the faithful flatness. $\diamond$

Corollary 3.8 *Sequence (3.10) is a strictly exact sequence of Hopf algebras.*

Corollary 3.9 *Sequence (3.10) allows one to view $SL_q(2)$ as a quantum group covering of $SL(2,\mathbb{C})$ (see Section 18 in [PW91]).*

Remark 3.10 We can think of $SL_q(2)$ as a quantum principal bundle over $SL(2,\mathbb{C})$. This bundle, however, is *not* locally trivial in the sense of [D-M96, p.460]. Indeed, otherwise it would have to be reducible to its classical subbundle (see p.466 in [D-M96]), which is impossible because $SL_q(2)$ has "less" classical points (characters of $A(SL_q(2))$) than $SL(2,\mathbb{C})$. (Cf. Section 4.2 in [BK96].) $\diamond$

4 Quotients of $A(SL_{e^{\frac{2\pi i}{3}}}(2))$ as cleft Hopf-Galois extensions

Let us now consider the case of (quantum) Borel subgroups. To abbreviate notation, in analogy with the previous section, we put $P_+ = A(SL_q(2))/\langle c\rangle$, $B_+ = A(SL(2,\mathbb{C}))/\langle\bar{c}\rangle$, and $H_+ = P_+/\langle a^3-1,\, b^3\rangle = A(F)/\langle\tilde{c}\rangle$. (We abuse the notation by not distinguishing formally generators of P, P_+, P_-, $P_\pm$, etc.) As in the previous section, we have the Frobenius homomorphism (cf. [PW91, Section 7.5]) $Fr_+ : B_+ \to P_+$ given by the same formula as (3.9), and the associated exact sequence of Hopf algebras:

$$B_+ \xrightarrow{Fr_+} P_+ \xrightarrow{\pi_+} H_+\,.$$

Before proceeding further, let us first establish a basis of P_+ and a basis of H_+ .

Proposition 4.1 *The set $\{a^p b^r\}_{p,r\in\mathbb{Z},\, r\geq 0}$ is a basis of P_+.*

Proof. This proof is based on the Diamond Lemma (Theorem 1.2 in [B-G78]). Let $\mathbb{C}\langle\alpha,\beta,\delta\rangle$ be the free unital associative algebra generated by α,β,δ. We well-order the monomials of $\mathbb{C}\langle\alpha,\beta,\delta\rangle$ first by their length, and then "lexicographically" choosing the following order among letters: $\alpha\preceq\delta\preceq\beta$. In particular, this is a semigroup partial ordering having descending chain condition, as required by the Diamond Lemma. Furthermore, we chose the reduction system $\mathcal{S}$ to be:

$$\mathcal{S} = \big\{(\alpha\delta,1)\,,\,(\delta\alpha,1)\,,\,(\beta\alpha,q^{-1}\alpha\beta)\,,\,(\beta\delta,q\delta\beta)\big\}\,.$$

It is straightforward to check that the aforementioned well-ordering is compatible with $\mathcal{S}$, there are no inclusion ambiguities in $\mathcal{S}$, and all overlap ambiguities of $\mathcal{S}$ are resolvable. Therefore, by the Diamond Lemma, the set of all $\mathcal{S}$-irreducible monomials is a basis of $\mathbb{C}\langle\alpha,\beta,\delta\rangle/J$, $J := \langle\alpha\delta-1\,,\,\delta\alpha-1\,,\,\beta\alpha-q^{-1}\alpha\beta\,,\,\beta\delta-q\delta\beta\rangle$. The monomials $\alpha^p\beta^r$, $\delta^k\beta^l$, $p,r,k,l\in\mathbb{N}_0$, $k>0$, are irreducible under $\mathcal{S}$ and their image under the canonical surjection spans $\mathbb{C}\langle\alpha,\beta,\delta\rangle/J$. Consequently, they form a basis of $\mathbb{C}\langle\alpha,\beta,\delta\rangle/J$. To conclude the proof it suffices to note that the algebras $\mathbb{C}\langle\alpha,\beta,\delta\rangle/J$ and P_+ are isomorphic. $\square$

Proposition 4.2 *The set* $\{\tilde{a}^p\tilde{b}^r\}_{p,r\in\{0,1,2\}}$ *is a basis of* H_+.

Proof. Analogous to the proof of Proposition 3.1. □

The formula for the right coaction of H_+ on P_+ is not as complicated as (3.8) and reads:

$$\Delta_R(a^p b^r) = \sum_{\mu=0}^{r} \binom{r}{\mu}_q q^{-\mu(2r-2\mu)} a^{p+\mu} b^{r-\mu} \otimes \tilde{a}^{2r+p-2\mu}\tilde{b}^{\mu} \,. \tag{4.12}$$

With the above formula at hand, it is a matter of a straightforward calculation to prove that P_+ is an H_+-Galois extension of $Fr_+(B_+)$. In particular, we have $P_+^{coH_+} = Fr_+(B_+)$. Moreover, since P_+ is generated by a group-like and a skew-primitive element, it is a pointed Hopf algebra. Consequently (see p.291 in [S-H92]), we obtain:

Proposition 4.3 P_+ *is a cleft* H_+*-Galois extension of* $Fr_+(B_+)$.

Our next step is to construct a family of cleaving maps for this extension. To simplify the notation, for the rest of this paper we will identify B_+ with its image under Fr_+. First, we construct a family of unital convolution invertible B_+-linear maps $\Psi_\nu : P_+ \to B_+$, and then employ Corollary 2.7. It is straightforward to verify that, for any function $\nu : \{0,1,2\} \to \mathbb{Z}$ satisfying $\nu(0) = 0$, the family $\{\Psi_\nu\}$ of B_+-homomorphisms given by the formula

$$\Psi_\nu(a^p b^r) := \delta_{0r} a^{3\nu(p)},\ p, r \in \{0,1,2\}, \tag{4.13}$$

fulfills the desired conditions. The convolution inverse of Ψ_ν is provided by (see [AD95, p.47])

$$\Psi_\nu^{-1}(a^p b^r) := \delta_{0r} a^{-3\nu(p)}, \quad p, r \in \{0,1,2\}, \quad \Psi_\nu^{-1}(wt) := \Psi_\nu^{-1}(t)S(w), \quad t \in P_+, \quad w \in B_+ \,.$$

Consequently,

$$\Phi_\nu : H_+ \to P_+,\ \Phi_\nu(\tilde{a}^p\tilde{b}^r) = \Psi_\nu^{-1}(a^{p+r})a^p b^r = a^{-3([p+r]_1+\nu([p+r]_2))+p} b^r, \tag{4.14}$$

where $3[p+r]_1 + [p+r]_2 = p+r,\ 0 \le [p+r]_2 < 3$, is a family of cleaving maps. In particular, we can choose $\nu(1) = 0,\ \nu(2) = 1$. Then we have:

$$\begin{aligned} &\Phi(1) = 1\ ,\ \Phi(\tilde{a}) = a\ ,\ \Phi(\tilde{a}^2) = a^{-1}\ ,\ \Phi(\tilde{b}) = b\ ,\ \Phi(\tilde{b}^2) = a^{-3}b^2\ , \\ &\Phi(\tilde{a}\tilde{b}) = a^{-2}b\ ,\ \Phi(\tilde{a}^2\tilde{b}) = a^{-1}b\ ,\ \Phi(\tilde{a}\tilde{b}^2) = a^{-2}b^2\ ,\ \Phi(\tilde{a}^2\tilde{b}^2) = a^{-1}b^2\ . \end{aligned} \tag{4.15}$$

Remark 4.4 Here we rely on the fact that the monomials $a^p b^r,\ p, r \in \{0,1,2\}$, form a B_+-basis of P_+. As can be proven with the help of the linear basis $\{a^p b^r\}_{p,r\in\mathbb{Z},\, r\ge 0}$, the set $\{a^p b^r\}_{p,r\in\{0,\ldots,n-1\}}$ is a B_+-basis of P_+ for any n-th primitive odd root of unity. Hence our construction of a family of cleaving maps can be immediately generalised to an arbitrary primitive odd root of unity. ◇

Let us now apply Lemma 1.9 to calculate explicitly the cocycle $\sigma_\Phi : H_+ \otimes H_+ \to B_+$:

$$\begin{array}{ccc}
\sigma_\Phi(\tilde{a}\otimes\tilde{a}) = a^3, & \sigma_\Phi(\tilde{a}^2\otimes\tilde{a}^2) = a^{-3}, & \sigma_\Phi(\tilde{b}\otimes\tilde{b}^2) = a^{-3}b^3, \\
\sigma_\Phi(\tilde{b}\otimes\tilde{a}\tilde{b}^2) = q^2a^{-3}b^3, & \sigma_\Phi(\tilde{b}\otimes\tilde{a}^2\tilde{b}^2) = qb^3, & \sigma_\Phi(\tilde{b}^2\otimes\tilde{b}) = a^{-3}b^3, \\
\sigma_\Phi(\tilde{b}^2\otimes\tilde{a}\tilde{b}) = qa^{-6}b^3, & \sigma_\Phi(\tilde{b}^2\otimes\tilde{a}^2\tilde{b}) = q^2a^{-3}b^3, & \sigma_\Phi(\tilde{a}\tilde{b}\otimes\tilde{b}^2) = a^{-6}b^3, \\
\sigma_\Phi(\tilde{a}\tilde{b}\otimes\tilde{a}\tilde{b}^2) = q^2a^{-3}b^3, & \sigma_\Phi(\tilde{a}\tilde{b}\otimes\tilde{a}^2\tilde{b}^2) = qa^{-3}b^3, & \sigma_\Phi(\tilde{a}^2\tilde{b}\otimes\tilde{b}^2) = a^{-3}b^3, \\
\sigma_\Phi(\tilde{a}^2\tilde{b}\otimes\tilde{a}\tilde{b}^2) = q^2a^{-3}b^3, & \sigma_\Phi(\tilde{a}^2\tilde{b}\otimes\tilde{a}^2\tilde{b}^2) = qa^{-3}b^3, & \sigma_\Phi(\tilde{a}\tilde{b}^2\otimes\tilde{b}) = a^{-3}b^3, \\
\sigma_\Phi(\tilde{a}\tilde{b}^2\otimes\tilde{a}\tilde{b}) = qa^{-3}b^3, & \sigma_\Phi(\tilde{a}\tilde{b}^2\otimes\tilde{a}^2\tilde{b}) = q^2a^{-3}b^3, & \sigma_\Phi(\tilde{a}^2\tilde{b}^2\otimes\tilde{b}) = b^3, \\
\sigma_\Phi(\tilde{a}^2\tilde{b}^2\otimes\tilde{a}\tilde{b}) = qa^{-3}b^3, & \sigma_\Phi(\tilde{a}^2\tilde{b}^2\otimes\tilde{a}^2\tilde{b}) = q^2a^{-3}b^3, & \sigma_\Phi|_{\text{other basis elements}} = \varepsilon\otimes\varepsilon\,.
\end{array} \tag{4.16}$$

As the cocycle action (see (1.2)) is necessarily trivial due to the centrality of B_+ in P_+, we obtain the following:

Proposition 4.5 *P_+ is isomorphic as a comodule algebra to the twisted product (see [BCM86, Example 4.10]) of B_+ with H_+ defined by the above described cocycle σ_Φ .*

More explicitly, we can simply say that the algebra structure on $B_+ \otimes H_+$ that is equivalent to the algebra structure of P_+ is given by the formula

$$(x \otimes h) \cdot (y \otimes l) = xy\sigma_\Phi(h_{(1)} \otimes l_{(1)}) \otimes h_{(2)}l_{(2)} \,. \tag{4.17}$$

Let us also mention that σ_Φ is *not* a coboundary, i.e., it cannot be gauged by a unital convolution invertible map $\gamma : H_+ \to B_+$ to the trivial cocycle $\varepsilon \otimes \varepsilon$ (see Proposition 6.3.4 in [M-S95]). More formally, we have:

Proposition 4.6 *The cocycle σ_Φ represents a non-trivial cohomology class in the (non-Abelian) 2-cohomology of H_+ with values in B_+ .*

Proof. Suppose that the claim of the proposition is false. Then there would exist γ such that $\sigma_{\gamma*\Phi} = \varepsilon \otimes \varepsilon$, i.e.,

$$[m \circ (\Phi^\gamma \otimes \Phi^\gamma)] * [(\Phi^\gamma)^{-1} \circ m] = \varepsilon \otimes \varepsilon \,. \tag{4.18}$$

Here $\Phi^\gamma := \gamma * \Phi$ and the middle convolution product is defined with respect to the natural coalgebra structure on $H_+ \otimes H_+$, namely $\Delta^\otimes := (id \otimes \mathrm{flip} \otimes id) \circ (\Delta \otimes \Delta)$. (Note that Φ^γ is also a cleaving map.) A standard argument (apply $*(\Phi^\gamma \circ m)$ from the right to both sides of (4.18)) allows us to conclude that Φ^γ is an algebra homomorphism. Since Φ^γ is injective (see Section 1), we can view H_+ as a subalgebra of P_+ . In particular, there exists $0 \neq p \in P_+$ such that $p^2 = 0$. (Put $p = \Phi^\gamma(\tilde{b}^2)$.) Write p as $\sum_{\mu\in\mathbb{Z}} a^\mu p_\mu$, where the coefficients $\{p_\mu\}_{\mu\in\mathbb{Z}}$ are polynomials in $\tilde{b}$. Let $\mu_0(p) := \max\{\mu \in \mathbb{Z} \,|\, p_\mu \neq 0\}$. It is well defined because $a^\mu b^n$, $\mu, n \in \mathbb{Z}$, $n \geq 0$, form a basis of P_+ , and exists because $p \neq 0$. Now, due to the commutation relation in P_+ and the fact that the polynomial ring $\mathbb{C}[b]$ has no zero divisors, we can conclude that $\mu_0(p^2)$ exists (and equals $2\mu_0(p)$). This contradicts the equality $p^2 = 0$. [3] □

To put it simply, H_+ cannot be embedded in P_+ as a subalgebra.

Remark 4.7 Note that we could equally well try to use the *lower* (quantum) Borel subgroups P_-, B_-, H_-. The Hopf algebras H_+ and H_- are naturally isomorphic as algebras and anti-isomorphic as coalgebras via the map that sends $\tilde{a}$ to $\tilde{a}$ and $\tilde{b}$ to $\tilde{c}$. They are also isomorphic as coalgebras and anti-isomorphic as algebras via the map that sends $\tilde{a}$ to $\tilde{a}^2$ and $\tilde{b}$ to $\tilde{c}$. It might be worth noticing that H_+ and H_- are *not* isomorphic as Hopf algebras. Indeed, if they were so, there would exist an invertible algebra map $\varphi : H_+ \to H_-$ commuting with the antipodes. From direct computations, it turns out that any such map has to satisfy $\varphi(\tilde{b}) = \kappa(\tilde{a} - q^2\tilde{a}^2)\tilde{c}^2$, with κ an arbitrary constant. This implies $\varphi(\tilde{b})^2 = \varphi(\tilde{b}^2) = 0$ contradicting, due to $\tilde{b}^2 \neq 0$ (see Proposition 4.2), the injectivity of φ . ◇

To end this section, let us consider the Cartan case: We define the Hopf algebras $P_\pm$, $B_\pm$ and $H_\pm$ by putting the off-diagonal generators to 0, i.e., $P_\pm := P/\langle b, c\rangle$, $B_\pm := B/\langle \bar{b}, \bar{c}\rangle$, $H_\pm := H/\langle \tilde{b}, \tilde{c}\rangle$. Everything is now commutative, and we have $P_\pm \cong B_\pm \cong A(\mathbb{C}^\times)$, $H_\pm \cong A(\mathbb{Z}_3)$, where $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. It is immediate to see that, just as in the above discussed Borel case, we have an exact sequence of Hopf algebras $B_\pm \overset{Fr_\pm}{\to} P_\pm \to H_\pm$, and $P_\pm$ is a cleft $H_\pm$-Galois extension of $Fr_\pm(B_\pm)$. A cleaving map Φ and cocycle σ_Φ are given by the formulas that look exactly as the a-part of (4.15) and (4.16) respectively. It might be worth to emphasize that, even though this extension is cleft, the principal bundle $\mathbb{C}^\times(\mathbb{C}^\times, \mathbb{Z}_3)$ is *not* trivial. Otherwise $\mathbb{C}^\times$ would have to be disconnected. This is why we call Φ a cleaving map rather than a trivialisation.

[3] We are grateful to Ioannis Emmanouil for helping us to make the nilpotent part of the proof simple.

5 The bicrossproduct structure of P_+

In particular, the concept of cocleftness applies to the Hopf-Galois extensions obtained from short exact sequences of Hopf algebras. One can view cocleftness as dual to cleftness the same way crossed coproducts are dual to crossed products [M-S90]. The upper Borel extension $B_+ \subseteq P_+$ is cleft and cocleft (see Corollary 2.7), and the maps Ψ_ν of (4.13) are both unital and counital. By Proposition 3.2.9 in [AD95], the cocleftness implies that P_+ is isomorphic as a left B_+-module coalgebra to the crossed coproduct of B_+ and H_+ given by the weak coaction

$$\lambda : H_+ \longrightarrow H_+ \otimes B_+ , \quad \lambda(\pi_+(p)) := \pi_+(p_{(2)}) \otimes \Psi^{-1}(p_{(1)})\Psi(p_{(3)}) ,$$

and the co-cocycle

$$\zeta : H_+ \longrightarrow B_+ \otimes B_+ , \quad \zeta(\pi_+(p)) := \Delta(\Psi^{-1}(p_{(1)}))(\Psi(p_{(2)}) \otimes \Psi(p_{(3)})) .$$

Here Ψ is the retraction obtained from (4.13) for the choice of ν made above (4.15). Explicitly, we have:

$$\begin{aligned} &\lambda(\tilde{a}^p) = \tilde{a}^p \otimes 1 \ , \ \lambda(\tilde{b}) = \tilde{b} \otimes 1 \ , \ \lambda(\tilde{a}\tilde{b}) = \tilde{a}\tilde{b} \otimes a^{-3} \ , \ \lambda(\tilde{a}^2\tilde{b}) = \tilde{a}^2\tilde{b} \otimes a^{-3} \ , \\ &\lambda(\tilde{b}^2) = \tilde{b}^2 \otimes a^{-6} \ , \ \lambda(\tilde{a}\tilde{b}^2) = \tilde{a}\tilde{b}^2 \otimes a^{-3} \ , \ \lambda(\tilde{a}^2\tilde{b}^2) = \tilde{a}^2\tilde{b}^2 \otimes a^{-3} \ . \end{aligned} \tag{5.19}$$

The co-cocycle is trivial, i.e., $\zeta(\pi_+(p)) = \varepsilon(p) \otimes 1$. We have thus arrived at:

Proposition 5.1 *P_+ is isomorphic as a left B_+-module coalgebra to the crossed coproduct ${B_+}^\lambda\# H_+$ defined by the above coaction λ.*

In particular, this means that the coproduct on $B_+ \otimes H_+$ that makes it isomorphic to P_+ as a coalgebra is given by

$$\Delta(w \otimes h) = w_{(1)} \otimes \lambda^{[1]}(h_{(1)}) \otimes w_{(2)}\lambda^{[2]}(h_{(1)}) \otimes h_{(2)} \ ,$$

where $\lambda(h) := \lambda^{[1]}(h) \otimes \lambda^{[2]}(h)$ (summation suppressed).

Proposition 5.2 *The above defined coproduct is* not *equivalent to the tensor coproduct $\Delta^\otimes := (id \otimes \mathrm{flip} \otimes id) \circ (\Delta \otimes \Delta) : B_+ \otimes H_+ \longrightarrow (B_+ \otimes H_+) \otimes (B_+ \otimes H_+)$.*

Proof. Suppose the contrary. Then, by [AD95, Proposition 3.2.12], there would exist a counital convolution invertible map $\xi : H_+ \to B_+$ such that $\lambda(h) = h_{(2)} \otimes \xi^{-1}(h_{(1)})\xi(h_{(3)})$. With the help of Proposition 4.2, applying this formula to $\tilde{b}$ implies $\xi^{-1}(\tilde{a})\xi(\tilde{a}^2) = 1$, and requiring it for $\tilde{b}^2$ gives $\xi^{-1}(\tilde{a}^2)\xi(\tilde{a}) = a^{-6}$. Since $\tilde{a}$ is group-like, $\xi(\tilde{a})$ and $\xi(\tilde{a}^2)$ are invertible, and we obtain $1 = \xi^{-1}(\tilde{a})\xi(\tilde{a}^2) = a^6$. This contradicts Proposition 4.1. □

Note that as far as the algebra structure of P_+ is concerned, it is given by the trivial action and a non-trivial cocycle. For the coalgebra structure it is the other way round, i.e., it is given by the trivial co-cocycle and a non-trivial coaction (cf. [M-S97]). Due to the triviality of co-cocycle ζ, Ψ is a coalgebra homomorphism. Also, one can check that the cocycle and coaction put together make $B_+ \otimes H_+$ a *cocycle bicrossproduct Hopf algebra* [M-S95].

Corollary 5.3 *The Hopf algebra P_+ is isomorphic to the cocycle bicrossproduct Hopf algebra ${B_+}^\lambda\#_\sigma H_+$. The isomorphism and its inverse are given by*

$$p \mapsto \Psi(p_{(1)}) \otimes \pi_+(p_{(2)}) \ , \quad w \otimes h \mapsto w\Phi(h) \ .$$

Here Φ is related to Ψ as in Corollary 2.7, and given explicitly by formulas (4.15).

6 Integrals on and in $A(F)$

Recall that a left (respectively right) integral *on* a Hopf algebra H over a field k is a linear functional $h : H \to k$ satisfying:

$$(id \otimes h) \circ \Delta = 1_H \cdot h \quad \text{(respectively } (h \otimes id) \circ \Delta = 1_H \cdot h). \tag{6.20}$$

(For a comprehensive review of the theory of integrals see [M-S95, Section 1.7], [S-M69, Chapter V].) In the case of the Hopf algebra $A(F)$, we have the following result:

Proposition 6.1 *The space of left integrals on $A(F)$ coincides with the space of right integrals on $A(F)$, i.e., $A(F)$ is a unimodular Hopf algebra. In terms of the basis $\{\tilde{a}^p\tilde{b}^r\tilde{c}^s\}_{p,r,s\in\{0,1,2\}}$ of $A(F)$, for any integral h, we have by $h(\tilde{a}^p\tilde{b}^r\tilde{c}^s) = z\delta_0^p\delta_2^r\delta_2^s$, $z \in \mathbb{C}$.*

Proof. By applying the projection $\pi_\pm : A(F) \to H_\pm$ to (6.20), it is easy to see that any left (and similarly any right) integral has to vanish on about half of the elements of the basis. With this information at hand, and using the fact that on a finite dimensional Hopf algebra the space of left and the space of right integrals are one dimensional [LS69], it is straightforward to verify by a direct calculation the claim of the proposition. □

A two-sided integral on a Hopf algebra H is called a *Haar measure* iff it is *normalised*, i.e., iff $h(1) = 1$. As integrals on $A(F)$ are *not* normalisable, we have:

Corollary 6.2 *There is no Haar measure on the Hopf algebra $A(F)$ (cf. Theorem 2.16 in [KP97] and (3.2) in [MMNNU91]).*

Remark 6.3 Since the Hopf algebra $A(F)$ is finite dimensional, F can be considered as a finite quantum group. However, it is *not* a compact matrix quantum group in the sense of Definition 1.1 in [W-S87]. Indeed, by Theorem 4.2 in [W-S87], compact matrix quantum groups always admit a (unique) Haar measure. Furthermore, as $A(F)$ satisfies all the axioms of Definition 1.1 in [W-S87] except for the C^*-axiom, there does not exist a $*$-structure and a norm on $A(F)$ that would make $A(F)$ a Hopf-C^*-algebra. In particular, for the $*$-structure given by setting $\tilde{a}^* = \tilde{a}$, $\tilde{b}^* = \tilde{b}$, $\tilde{c}^* = \tilde{c}$, $\tilde{d}^* = \tilde{d}$, this fact is evident: Suppose that there exists a norm satisfying the C^*-conditions. Then $0 = \parallel \tilde{c}^4 \parallel = \parallel (\tilde{c}^2)^*\tilde{c}^2 \parallel = \parallel \tilde{c}^2 \parallel^2$, which implies $\tilde{c}^2 = 0$ and thus contradicts Proposition 3.1. ◇

We recall also that an element $\Lambda \in H$ is called a left (respectively right) integral *in* H, iff it verifies $\alpha\Lambda = \varepsilon(\alpha)\Lambda$, (respectively $\Lambda\alpha = \varepsilon(\alpha)\Lambda$) for any $\alpha \in H$. If H is finite dimensional, an integral in H corresponds to an integral on the dual Hopf algebra H^*. Clearly, an integral in $A(F)$ should annihilate any non-constant polynomial in $\tilde{b}$ and $\tilde{c}$, whereas it should leave unchanged any polynomial in $\tilde{a}$. It is easy to see that the element $\Lambda_L = (1 + \tilde{a} + \tilde{a}^2)\tilde{b}^2\tilde{c}^2$ is a left integral and the element $\Lambda_R = \tilde{b}^2\tilde{c}^2(1 + \tilde{a} + \tilde{a}^2)$ is a right integral. Hence in this case left and right integrals are not proportional. We can therefore conclude that H^*, which by Section 3 in [DNS97] can be identified with $U_q(sl_2)/\langle K^3-1, E^3, F^3\rangle$ of [C-R97], is *not* unimodular. Again, since $A(F)$ is finite dimensional, any left integral in $A(F)$ is proportional to Λ_L, and any right integral in $A(F)$ is proportional to Λ_R. In addition, by Theorem 5.1.8 in [S-M69], the property $\varepsilon(\Lambda_L) = 0$ assures us that $A(F)$ is *not* semisimple as an algebra.

7 A coaction of $A(F)$ on $M(3,\mathbb{C})$

Let us now consider F as a quantum-group symmetry of $M(3,\mathbb{C})$ — a direct summand of A. Connes' algebra for the Standard Model. Recall first that for any $n \in \mathbb{N}$ the algebra of matrices $M(n,\mathbb{C})$ can be identified with the algebra $\mathbb{C}\langle \mathrm{x}, \mathrm{y}\rangle / \langle \mathrm{xy} - \mu \mathrm{yx},\, \mathrm{x}^n - 1,\, \mathrm{y}^n - 1\rangle$, $\mu = e^{\frac{2\pi i}{n}}$. (Map x to $\left(\begin{smallmatrix} 0 & I_{n-1} \\ 1 & 0 \end{smallmatrix}\right)$ and y to $diag(1, \mu, ..., \mu^{n-1})$; see Section IV.D.15 of [W-H31].) Denoting by $A(\mathbb{C}^2_q)$ the polynomial algebra (in x and y) of the quantum plane, by $A(\mathbb{C}^2) = \mathbb{C}[\bar{x}, \bar{y}]$ the algebra of polynomials on $\mathbb{C}^2$, and maintaining our assumption that $q = e^{\frac{2\pi i}{3}}$, we obtain the following sequence of algebras and algebra homomorphisms:

$$A(\mathbb{C}^2) \xrightarrow{fr} A(\mathbb{C}^2_q) \xrightarrow{\pi_M} M(3,\mathbb{C}) \cong A(\mathbb{C}^2_q)/\langle x^3 - 1,\, y^3 - 1\rangle\ . \tag{7.21}$$

Here fr is an injection given by $fr(\bar{x}) = x^3$, $fr(\bar{y}) = y^3$, and π_M is the map induced by the canonical surjection $A(\mathbb{C}^2_q) \to A(\mathbb{C}^2_q)/\langle x^3 - 1,\, y^3 - 1\rangle$.

Let us note that although $A(\mathbb{C}^2_q)$ and $A(\mathbb{C}^2) \otimes M(3,\mathbb{C})$ are isomorphic as $A(\mathbb{C}^2)$-modules, their algebraic structures (cf. (4.17)) are slightly different:

$$(\bar{x}^p \bar{y}^r \otimes \tilde{x}^k \tilde{y}^\ell)(\bar{x}^s \bar{y}^t \otimes \tilde{x}^m \tilde{y}^n) = \bar{x}^{p+s+[k+m]_1} \bar{y}^{r+t+[\ell+n]_1} \otimes \tilde{x}^{[k+m]_2} \tilde{y}^{[\ell+n]_2}\,, \tag{7.22}$$

where $3[n]_1 + [n]_2 = n$, $[n]_1, [n]_2 \in \mathbb{N}$, $0 \leq [n]_2 < 3$, $\tilde{x} = \pi_M(x)$, $\tilde{y} = \pi_M(y)$. Incidentally, the associativity of this product amounts to the identity $[k{+}m]_1 + [[k{+}m]_2{+}u]_1 = [m{+}u]_1 + [[k{+}[m{+}u]_2{+}u]_1$.

Next, observe that combining sequences (7.21) and (3.10) together with the natural right coactions ($\mathrm{e}_i \mapsto \sum_{j\in\{1,2\}} \mathrm{e}_j \otimes M_{ji}\,,\ i \in \{1,2\}$) on $A(\mathbb{C}^2)$, $A(\mathbb{C}^2_q)$, and $M(3,\mathbb{C})$ respectively, one can obtain the following commutative diagram of algebras and algebra homomorphisms:

$$\begin{array}{ccc}
A(\mathbb{C}^2) & \xrightarrow{\ \rho\ } & A(\mathbb{C}^2) \otimes A(SL(2,\mathbb{C})) \\
{\scriptstyle fr}\big\downarrow & & \big\downarrow{\scriptstyle fr\otimes Fr} \\
A(\mathbb{C}^2_q) & \xrightarrow{\ \rho_q\ } & A(\mathbb{C}^2_q) \otimes A(SL_q(2,\mathbb{C})) \\
{\scriptstyle \pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi_M\otimes\pi_F} \\
M(3,\mathbb{C}) & \xrightarrow{\ \rho_F\ } & M(3,\mathbb{C}) \otimes A(F)\ .
\end{array} \tag{7.23}$$

Another way to look at $M(3,\mathbb{C}) \overset{\rho_F}{\to} M(3,\mathbb{C}) \otimes A(F)$ is to treat $M(3,\mathbb{C})$ as a 9-dimensional comodule rather than a comodule algebra. Let us choose the following linear basis of $M(3,\mathbb{C})$:

$$e_1 = 1,\ e_2 = \tilde{x},\ e_3 = \tilde{y},\ e_4 = \tilde{x}^2,\ e_5 = \tilde{x}\tilde{y},\ e_6 = \tilde{y}^2,\ e_7 = \tilde{x}^2\tilde{y},\ e_8 = \tilde{x}\tilde{y}^2,\ e_9 = \tilde{x}^2\tilde{y}^2.$$

The formula $\Delta_R e_i = e_j \otimes N_{ji}$ allows us to determine the corepresentation matrix N:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \tilde{a}^2(\tilde{b} + q^2\tilde{c}^2) & \tilde{a}(\tilde{b}^2 + q^2\tilde{c} - q\tilde{b}\tilde{c}^2) & 0 \\
0 & \tilde{a} & \tilde{b} & 0 & 0 & 0 & 0 & 0 & \tilde{a}^2(\tilde{b}^2 - q\tilde{c}) \\
0 & \tilde{c} & \tilde{d} & 0 & 0 & 0 & 0 & 0 & \tilde{a}(q^2\tilde{b}^2\tilde{c} + q\tilde{c}^2 - \tilde{b}) \\
0 & 0 & 0 & \tilde{a}^2 & \tilde{a}\tilde{b} & \tilde{b}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -q^2\tilde{a}\tilde{c} & (1 - \tilde{b}\tilde{c}) & -q^2\tilde{b}\tilde{d} & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{c}^2 & \tilde{c}\tilde{d} & \tilde{d}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{a} & -\tilde{b} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\tilde{c} & \tilde{d} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{7.24}$$

It is clear that N is reducible. The upper right corner terms of N appear to be an effect of the finiteness of F. By restricting the comodule $M(3,\mathbb{C})$ respectively to the linear span of $1, \tilde{x}^2\tilde{y}, \tilde{x}\tilde{y}^2$ and the linear span of $\tilde{x}, \tilde{y}, \tilde{x}^2\tilde{y}^2$, we obtain two "exotic" corepresentations of $A(F)$ (see [DNS97, Section 4] for the dual picture):

$$N_1 = \begin{pmatrix} 1 & \tilde{a}^2(\tilde{b}+q^2\tilde{c}^2) & \tilde{a}(\tilde{b}^2+q^2\tilde{c}-q\tilde{b}\tilde{c}^2) \\ 0 & \tilde{a} & -\tilde{b} \\ 0 & -\tilde{c} & \tilde{d} \end{pmatrix}, \quad N_2 = \begin{pmatrix} \tilde{a} & \tilde{b} & \tilde{a}^2(\tilde{b}^2-q\tilde{c}) \\ \tilde{c} & \tilde{d} & \tilde{a}(q^2\tilde{b}^2\tilde{c}+q\tilde{c}^2-\tilde{b}) \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.25}$$

To end with, let us remark that, very much like the Frobenius map Fr, the "Frobenius-like" map fr of sequence (7.21) allows us to identify $A(\mathbb{C}^2)$ with the subalgebra of $(id\otimes\pi_F)\circ\rho_q$-coinvariants of $A(\mathbb{C}_q^2)$:

$$fr(A(\mathbb{C}^2)) = A(\mathbb{C}_q^2)^{coA(F)}. \tag{7.26}$$

Indeed, since we can embed $A(\mathbb{C}_q^2)$ in $A(SL_q(2))$ as a subcomodule algebra (e.g., $x\mapsto a$, $y\mapsto b$), equality (7.26) follows directly from Proposition 3.4 and the lemma below:

Lemma 7.1 *Let P_1 and P_2 be right H-comodules, and $\mathrm{j}: P_1\to P_2$ an injective comodule homomorphism. Then $P_1^{coH} = \mathrm{j}^{-1}(P_2^{coH})$.*

Proof. Denote by $\rho_1 : P_1 \to P_1\otimes H$ and $\rho_2 : P_2\to P_2\otimes H$ the right H-coactions on P_1 and P_2 respectively. Assume now that $p\in P_1^{coH}$. Then $\rho_2(\mathrm{j}(p)) = (\mathrm{j}\otimes id)(\rho_1(p)) = \mathrm{j}(p)\otimes 1$, i.e., $p\in \mathrm{j}^{-1}(P_2^{coH})$. Conversely, assume that $p\in \mathrm{j}^{-1}(P_2^{coH})$. Then $(\mathrm{j}\otimes id)(p\otimes 1) = \rho_2(\mathrm{j}(p)) = (\mathrm{j}\otimes id)(\rho_1(p))$. Consequently, by the injectivity of $(\mathrm{j}\otimes id)$, we have $\rho_1(p) = p\otimes 1$, i.e., $p\in P_1^{coH}$. $\square$

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